

A RELAXATION OF STEINBERG'S CONJECTURE

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ABSTRACT. A graph is (c_1, c_2, \dots, c_k) -colorable if the vertex set can be partitioned into k sets V_1, V_2, \dots, V_k , such that for every $i : 1 \leq i \leq k$ the subgraph $G[V_i]$ has maximum degree at most c_i . We show that every planar graph without 4- and 5-cycles is $(1, 1, 0)$ -colorable and $(3, 0, 0)$ -colorable. This is a relaxation of the Steinberg Conjecture that every planar graph without 4- and 5-cycles are properly 3-colorable (i.e., $(0, 0, 0)$ -colorable).

1. INTRODUCTION

It is well-known that the problem of deciding whether a planar graph is properly 3-colorable is NP-complete. Grötzsch in 1959 [5] showed the famous theorem that every triangle-free planar graph is 3-colorable. A lot of research was devoted to find sufficient conditions for a planar graph to be 3-colorable, by allowing a triangle together with some other conditions. One of such efforts is the following famous conjecture made by Steinberg in 1976.

Conjecture 1 (Steinberg, [7]). *All planar graphs without 4-cycles and 5-cycles are 3-colorable.*

Not much progress in this direction was made until Erdős proposed to find a constant C such that a planar graph without cycles of length from 4 to C is 3-colorable. Borodin, Glebov, Raspaud, and Salavatipour [2] showed that $C \leq 7$. For more results, see the recent nice survey by Borodin [1].

Yet another direction of relaxation of the Conjecture is to allow some defects in the color classes. A graph is (c_1, c_2, \dots, c_k) -colorable if the vertex set can be partitioned into k sets V_1, V_2, \dots, V_k , such that for every $i : 1 \leq i \leq k$ the subgraph $G[V_i]$ has maximum degree at most c_i . Thus a $(0, 0, 0)$ -colorable graph is properly 3-colorable.

Eaton and Hull [4] and independently Škrekovski [6] showed that every planar graph is $(2, 2, 2)$ -colorable (actually choosable). Xu [8] proved that all planar graphs with no adjacent triangles or 5-cycles are $(1, 1, 1)$ -colorable. Chang, Havet, Montassier, and Raspaud [3] proved that all planar graphs without 4-cycles or 5-cycles are $(2, 1, 0)$ -colorable and $(4, 0, 0)$ -colorable. In this paper, we further prove the following relaxation of the Steinberg Conjecture.

Theorem 1. *All planar graphs without 4-cycles and 5-cycles are $(1, 1, 0)$ -colorable.*

Theorem 2. *All planar graphs without 4-cycles and 5-cycles are $(3, 0, 0)$ -colorable.*

We will use a discharging argument in the proofs. We let the initial charge of vertex $u \in G$ be $\mu(u) = 2d(u) - 6$, and the initial charge of face f be $\mu(f) = d(f) - 6$. Then by Euler's formula, we have

$$(1) \quad \sum_{v \in V(G)} \mu(v) + \sum_{f \in F(G)} \mu(f) = -12.$$

Our goal is to show that we may re-distribute the charges among vertices and faces so the final charges of the vertices and faces are non-negative, which would be a contradiction. In the process of discharging, we will see that some configurations prevent us from showing some vertices or faces to have non-negative charges. Those configurations will be shown to be reducible configurations, that is, a valid coloring outside of the configurations can be extended to the whole graph. It is worth to note that in the proof of Theorem 1, we prove a somewhat global structure, a special chain of triangles, to be reducible.

The following are some simple observations about the minimal counterexamples to the above theorems.

Proposition 1. *Among all planar graphs without 4-cycles and 5-cycles that are not $(1, 1, 0)$ -colorable or $(3, 0, 0)$ -colorable, let G be one with minimum number of vertices. Then*

- (a) G contains no 2^- vertices.
- (b) a k -vertex in G can have $\alpha \leq \lfloor \frac{k}{2} \rfloor$ incident 3-faces, and at most $k - 2\alpha$ pendant 3-faces.

We will use the following notations in the proofs. A k -vertex (k^+ -vertex, k^- -vertex) is a vertex of degree k (at least k , at most k resp.). The same notation will apply to faces. An $(\ell_1, \ell_2, \dots, \ell_k)$ -face is a k -face with incident vertices of degree $\ell_1, \ell_2, \dots, \ell_k$. A *bad 3-vertex* is a 3-vertex on a 3-face. A face f is a *pendant 3-face* to vertex v if v is adjacent to some bad 3-vertex on f . The *pendant neighbor* of a 3-vertex v on a 3-face is the neighbor of v not on the 3-face. A vertex v is *properly colored* if all neighbors of v have different colors from v . A vertex v is *nicely colored* if it shares colors with at most $\max\{s_i - 1, 0\}$ neighbors, thus if a vertex v is nicely colored by a color c which allows deficiency $s_i > 0$, then an uncolored neighbor of v can be colored by c .

In the next section, we will give a proof to Theorem 1; and in the last section, we will give a proof to Theorem 2.

2. $(1, 1, 0)$ -COLORING OF PLANAR GRAPHS

We will use a discharging argument in our proof. First we will prove some reducible configurations.

Let G be a minimum counterexample to Theorem 1, that is, G is a planar graph without 4-cycles and 5-cycles, and G is not $(1, 1, 0)$ -colorable, but any proper subgraph of G is $(1, 1, 0)$ -colorable.

The following is a very useful tool in the proofs.

Lemma 1. *Let H be a proper subgraph of G so that there is a $(1, 1, 0)$ -coloring of $G - H$. If vertex $v \in H$ satisfies either (i) 3 neighbors of v are colored, with at least two properly colored, or (ii) 4 neighbors of v are colored, all properly, then the coloring of $G - H$ can be extended to $G - (H - v)$.*

Proof. (i) Let $v \in H$ be a vertex with 3 colored neighbors, two of which are properly colored, such that the coloring of $G - H$ can not be extended to v . Since v is not $(1, 1, 0)$ -colorable, the three neighbors of v must have different colors, and furthermore, two of the colored neighbors cannot be properly colored, a contradiction to the assumption that two of the colored neighbors of v are properly colored.

(ii) Let $v \in H$ be a vertex of degree 4 with all neighbors properly colored such that the coloring of $G - H$ can not be extended to v . Then due to the coloring deficiencies, v must have at least 2 neighbors colored by 1, at least 2 neighbors colored by 2, and at least 1 neighbor colored by 1. Then v has at least five colored neighbors, a contradiction. \square

Lemma 2. *There is no $(3, 3, 4^-)$ -face in G .*

Proof. Let uvw be a $(3, 3, 4^-)$ -face in G with $d(u) = d(v) = 3$ and $d(w) \leq 4$. Then $G \setminus \{u, v, w\}$ is $(1, 1, 0)$ -colorable. Color w and v properly, then u is colorable by Lemma 1, thus G is $(1, 1, 0)$ -colorable, a contradiction. \square

Lemma 3. *There is no 5-vertex that is incident to two $(3, 4^-, 5)$ -faces and adjacent to a 3-vertex in G .*

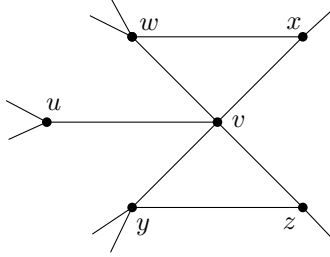


FIGURE 1. Figure for Lemma 3

Proof. Let v be a 5-vertex with neighbors u, w, x, y, z so that $w, x, y, z \in E(G)$ and $d(u) = d(x) = d(z) = 3$ and $d(w), d(y) \leq 4$ (See Figure 1). By the minimality of G , $G \setminus \{u, v, w, x, y, z\}$ is $(1, 1, 0)$ -colorable. Properly color u, w , and y , then properly color x and z . For v to not be colorable, v must have two neighbors colored by 1, two neighbors colored by 2 and one neighbor colored by 3. Since the w, x and y, z vertex pairs must be colored differently, one of them must have the colors 1 and 2. W.l.o.g. we can assume that w is colored by 1 and x by 2. Then since w is properly colored, we can either recolor x by 1 or 3, and color v by 2 obtaining a coloring of G , a contradiction. \square

Lemma 4. *No 3-vertex in G can be adjacent to two other 3-vertices. In particular, the 3-vertices on a $(3, 3, 5^+)$ -face must have another neighbor with degree four or higher.*

Proof. Let v be a 3-vertex with x and y being two neighbors of degree 3. By the minimality of G , $G \setminus \{v, x, y\}$ is $(1, 1, 0)$ -colorable. Then we can first properly color x and y , and then by Lemma 1 color v to get a coloring of G , a contradiction. \square

Lemma 5. *The pendant neighbor of the 3-vertex on a $(3, 4, 4)$ -face must have degree 4 or higher.*

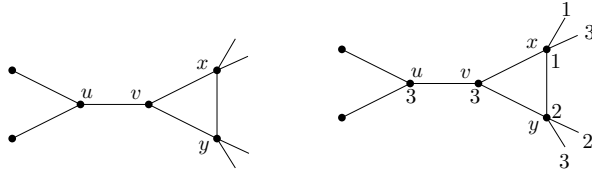


FIGURE 2. Figure for Lemma 5

Proof. Let vxy be a $(3, 4, 4)$ -face in G such that the pendant neighbor u of the 3-vertex v has degree 3 (See Figure 2). By the minimality of G , $G \setminus \{u, v\}$ is $(1, 1, 0)$ -colorable. We properly color u and then color v differently from both x and y . If u and v are not both colored by 3, then we get a coloring for G , a contradiction, so we may assume both u and v are colored by 3. This means that both u and v have two remaining neighbors colored by 1 and 2. Let x and y be colored by 1 and 2 respectively. The neighbors of x must be colored by 1 and 3 or else we could recolor v by 1 and x by 3 if necessary to obtain a coloring of G . Likewise, the neighbors of y must be colored by 2 and 3. In this case we switch the colors of x and y and color v by 1 to obtain a coloring of G , a contradiction again. \square

Let a (T_0, T_1, \dots, T_n) -chain be a sequence of triangles, T_0, T_1, \dots, T_n , such that (i) T_0 is a $(3, 4, 4)$ -face and T_n is a $(3^+, 4, 4^+)$ -face, and all other triangles are $(4, 4, 4)$ -faces, and (ii) for $0 \leq i \leq n-1$, T_i and T_{i+1} share a 4-vertex t_i . In a (T_0, T_1, \dots, T_n) -chain, let $x_i \in T_i$ for $0 \leq i \leq n$ be a non-connecting 4^+ -vertex.

Let a *special 4-vertex* be a 4-vertex that is incident to one 3-face and has two pendant 3-faces, and let a 3-face be a *special 3-face* if it has at least one special 4-vertex. Let a *good 4-vertex* be a 4-vertex with only one incident 3-face and at most one pendant 3-face.

We will prove in the following lemmas that a $(3, 4, 4)$ -face T_0 may get help in discharging from a $(3^+, 4^+, 5^+)$ -face or special 3-face T_n through a (T_0, T_1, \dots, T_n) -chain.

Lemma 6. *There are no special $(3, 4, 4)$ -faces in G .*

Proof. Let uvw be a special $(3, 4, 4)$ -face in G such that $d(v) = d(w) = 4$. W.l.o.g. we can assume that v is a special 4-vertex with pendant neighbors v_1 and v_2 . By the minimality of G , $G \setminus \{u, v, v_1, v_2, w\}$ is $(1, 1, 0)$ -colorable. We can properly color w and u in that order then properly color v_1 and v_2 . Then by Lemma 1, we can color u , obtaining a coloring of G , a contradiction. \square

The following is a very useful tool in extending a coloring to a chain.

Lemma 7. *Consider a (T_0, T_1, \dots, T_n) -chain with $n \geq 1$ and T_n being a $(4, 4^-, k)$ -face. If $G \setminus \{T_0, T_1, \dots, T_{n-1}\}$ has a coloring such that the k -vertex of T_n is properly colored, or it shares the same color with the 4^- -vertex, then the coloring can be extended to G .*

Proof. We assume that the $(4, 4^-, k)$ -face T_n has k -vertex x_n and 4^- -vertex t_n . Also let $G \setminus \{T_0, T_1, \dots, T_{i-1}\}$ has a coloring such that x_n is properly colored or shares the same color with t_n and G does not have a $(1, 1, 0)$ -coloring. Finally let u be the 3-vertex of T_0 and let w be the pendant neighbor of u .

We consider two cases. First let $n = 1$. If x_1 and t_1 have the same color, then we can properly color x_0 and t_0 in that order, thus by Lemma 1 we can color u so G has a $(1, 1, 0)$ -coloring, a contradiction. So we know that x_1 and t_1 must be colored differently, and further x_1 is colored properly. We can properly color x_0 . If x_0 and w share the same color then we can color t_0 by Lemma 1 and properly color u , a contradiction. So we may assume that x_0 and w are colored differently. If any two of x_0, x_1 , and t_1 are colored the same then we could color t_0 properly and color u by Lemma 1, a contradiction. Since x_0, x_1 , and t_1 are colored differently, if x_0 is not colored by 3 then we could color t_0 by the same color as x_0 and properly color u , a contradiction. So x_0 must be colored by 3 and w.l.o.g. we can assume that w is colored by 1. Since x_1 is properly colored, it must be colored by 2, or we could color t_0 by 1 and properly color u , a contradiction. It follows that t_1 is colored by 1. If t_1 is colored properly, then we could color t_0 by 1 and properly color u , a contradiction, so we may

assume that t_1 is not colored properly. Further, neither z nor z' (the two other neighbors of t_1) can be colored by 2, or we could recolor t_1 properly, then color t_0 by 1 and u properly, a contradiction. So we color t_1 by 2 and t_0 by 1, and properly color u , a contradiction.

Now we assume that $n \geq 2$. For all $j : 1 \leq j \leq n$, properly color x_{n-j} and color t_{n-j} by Lemma 1, or properly if possible. Then since x_1 was properly colored, and t_1 was colored after x_1 , either x_1 remains properly colored, or t_1 has the same color as x_1 . Also, we know that T_1 must be a $(4, 4, 4)$ -face, so by the previous case, we can extend the coloring to T_0 and get a coloring of G , a contradiction. \square

Lemma 8. *There is no (T_0, \dots, T_n) -chain so that (i) $n \geq 1$ and T_n is a special $(4, 4, 4)$ -face or (ii) $n \geq 2$ and T_n is a $(3, 4, k)$ -face or (iii) $n = 1$ and T_n is a $(3, 4, 4^-)$ -face.*

Proof. Let $T_0 = ux_0t_0$ be a $(3, 4, 4)$ -face with $d(u) = 3$.

(i) Let v be a special 4-vertex of T_n and let y and z be the neighbors of v other than t_n and x_n . Let $S = \{t_i, x_i : 0 \leq i \leq n-1\}$. By the minimality of G , $G \setminus (S \cup \{u, v, x_n, y, z\})$ has a $(1, 1, 0)$ -coloring. Properly color x_n , y and z , then by Lemma 1 color v . Then, either x_n remains properly colored or v shares the same color, so by Lemma 7 we can extend the coloring to $\{T_0, T_1, \dots, T_{n-1}\}$ to obtain a coloring of G .

(ii) Let v be the 3-vertex of T_n and let $S = \{t_i, x_i : 0 \leq i \leq n-1\}$. By the minimality of G , $G \setminus (S \cup \{u, v\})$ has a $(1, 1, 0)$ coloring. Properly color v and x_{n-1} . Then by Lemma 1, we can color t_{n-1} . Either x_{n-1} remains properly colored or t_{n-1} shares the same color, so by Lemma 7 we can extend the coloring to $\{T_0, T_1, \dots, T_{n-2}\}$ to obtain a coloring of G .

(iii) Assume that $n = 1$ and T_n is a $(3, 4, 4)$ -face with 3-vertex v . By the minimality of G , $G \setminus \{t_0, u, v, x_0, x_1\}$ has a $(1, 1, 0)$ -coloring. Properly color x_0 and u in that order and properly color x_1 and v in that order. Then t_0 has four neighbors colored, all properly, so by Lemma 1 we can color t_0 to get a coloring for G . \square

Remark: By above lemma, a (T_0, T_1) -chain with T_1 being a $(3, 4, 5^+)$ -face is not necessarily reducible. Let a *bad* $(3, 4, 5^+)$ -face be a $(3, 4, 5^+)$ -face that shares a 4-vertex with a $(3, 4, 4)$ -face.

Lemma 9. *There is no (T_0, \dots, T_n) -chain with $T_i = T_n$ for some $i \neq n$.*

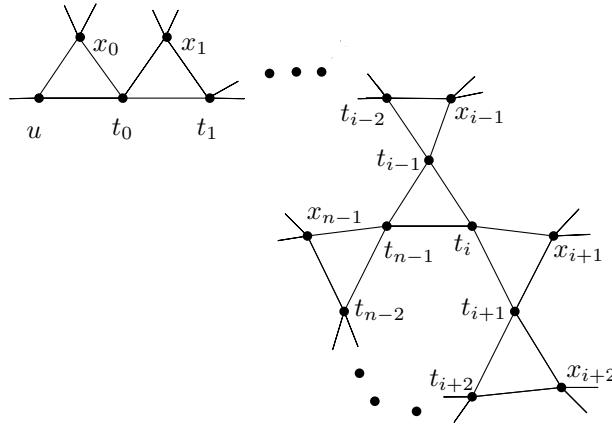


FIGURE 3. Figure for Lemma 9

Proof. Let (T_0, \dots, T_n) -chain be a chain with $T_i = T_n$ for some $i < n$. Let u be the 3-vertex of T_0 and let $S = \{t_j, x_j : 0 \leq j \leq n-1\}$. Since $T_i = T_n$, the vertex that would have been labelled x_i is instead labelled t_{n-1} (See Figure 3). By the minimality of G , $G \setminus (S \cup \{u\})$ is $(1, 1, 0)$ -colorable. Start by properly coloring x_{i+1} , x_{i+2} , and t_{i+1} . Then for all $j : i+2 \leq j \leq n-2$, properly color x_{j+1} and color t_j by Lemma 1. Next, properly color t_{n-1} , and we have two cases:

Case 1: $i = 0$. We can properly color u , then color t_i by Lemma 1 to get a coloring of G , a contradiction.

Case 2: $i > 0$. We can then color t_i by Lemma 1 and then either t_{n-1} is properly colored, or t_i shares the same color, so by Lemma 7 we can extend the coloring to $\{T_0, T_1, \dots, T_{i-1}\}$ to obtain a coloring of G , a contradiction. \square

Lemma 10. *For each $(3, 4, 4)$ -face T_0 without good 4-vertices, there exist two chains, (T_0, \dots, T_n) -chain and (T_0, \dots, T'_m) -chain, such that T_n and T'_m are either bad $(3, 4, 5^+)$ -faces, $(4, 4^+, 5^+)$ -faces, or $(4, 4, 4)$ -faces with a good 4-vertex. Furthermore, $T_n \neq T'_m$.*

Proof. As G is finite, any chain of triangles in G must be finite. By Lemma 8 and 9, no chain of triangles in G can end with a special 3-face or a non-bad $(3, 4, 5^+)$ -face, thus it must end with a bad $(3, 4, 5^+)$ -face or a $(4, 4^+, 4^+)$ -face. Since a $(4, 4, 4)$ -face in a chain can not be a special 3-face, any chain of triangles in G must end with a bad $(3, 4, 5^+)$ -face, a $(4, 4^+, 5^+)$ -face or a $(4, 4, 4)$ -face with a good 4-vertex.

Now we assume that $T_n = T'_m$. Then by Lemma 7, T_n must be a $(4, 4, 5^+)$ -face, and since G has no 4- and 5-cycles, $n + m \geq 6$. Assume that $n \leq m$. Let $S = \{t_i, x_i : 1 \leq i \leq n-1\}$, where $S = \emptyset$ if $n = 1$, and $S' = \{t'_j, x'_j : 0 \leq j \leq m-1\}$ and let u be the 3-vertex of T_0 . By the minimality of G , $G \setminus S \cup S' \cup \{u\}$ has a $(1, 1, 0)$ -coloring. We have two cases:

If $n = 1$, properly color x'_{m-1} and t'_{m-1} . Then, by Lemma 7 we can extend the coloring to $\{T_0, T'_1, \dots, T'_{m-2}\}$ to obtain a coloring of G , a contradiction.

If $n \geq 2$, then properly color x_{n-1} , t_{n-1} and x'_{m-1} in that order, then by Lemma 1 we can color t'_{m-1} . If $n \geq 3$, for all $i : 2 \leq i \leq n-1$, properly color x_{n-i} and by Lemma 1 we can color t_{n-i} . Then since either x'_{m-1} is still properly colored or shares the same color as t'_{m-1} , by Lemma 7 we can extend the coloring to $\{T_0, T'_1, \dots, T'_{m-2}\}$ to obtain a coloring of G , a contradiction. \square

We will now prove some lemmas which will ensure that bad $(3, 4, 5^+)$ -faces will have extra charge to help $(3, 4, 4)$ -faces.

Lemma 11. *A 5-vertex incident to a bad $(3, 4, 5)$ -face cannot be incident to another bad $(3, 4, 5)$ -face or a $(3, 3, 5)$ -face.*

Proof. We only show the case when a 5-vertex v is incident to two bad $(3, 4, 5)$ -faces, and it is very similar (and easier!) to show the case when it is incident to a bad $(3, 4, 5)$ -face and a $(3, 3, 5)$ -face.

Let v be a 5-vertex that is incident two bad $(3, 4, 5)$ -faces, f_1 and f_2 , and let u be a k -vertex adjacent v (see Figure 4). Let f_3 be the $(3, 4, 4)$ -face sharing a 4-vertex with f_1 and let f_4 be the $(3, 4, 4)$ -face sharing a 4-vertex with f_2 . Let f_3 and f_4 have outer 4-vertices of x and x' respectively and 3-vertices of y and y' respectively. Also, let f_1 and f_2 have 4-vertices z and z' . Then, by the minimality of G , $G \setminus \{f_1, f_2, f_3, f_4\}$ has a $(1, 1, 0)$ -coloring.

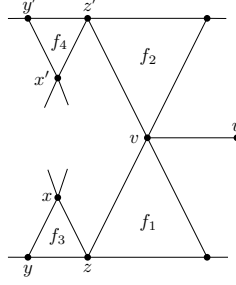


FIGURE 4. Figure for Lemma 11

If u is colored by 1 or 2, then we can color v by 3 and color the 3-vertices of f_1 and f_2 properly. Since v is properly colored, by Lemma 7 we can extend the coloring to f_1 and f_3 . Then, since v is colored by 3, it would remain properly colored, so again by Lemma 7 we can extend the coloring to f_2 and f_4 to get a coloring of G .

If u is colored by 3, then we properly color x and x' then properly color y and y' . We then properly color z and z' . If either z or z' is colored by 3, then we can properly color the 3-vertices of f_1 and f_2 and color v by either 1 or 2 getting a coloring for G . So we can assume neither is colored by 3, and w.l.o.g. we can assume that z is colored by 1. Then since z and z' are properly colored, we can color the 3-vertices of f_1 and f_2 by either 1 or 3. Then since v will have at most one neighbor colored by 2, and that neighbor colored properly, we can color v by 2 to obtain a coloring for G . \square

Lemma 12. *A $(3, 5, k)$ -face in G that is incident a 5-vertex that is also incident to a bad $(3, 4, 5)$ -face and a pendant $(3, 4^-, 4^-)$ -face will have a pendant neighbor that is a 4^+ -vertex.*

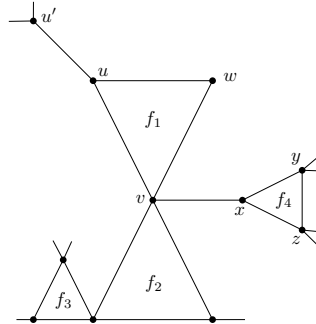


FIGURE 5. Figure for Lemma 12

Proof. Let f_1 be a $(3, 5, k)$ -face in G with a 5-vertex v , a 3-vertex u , and a pendant neighbor u' that is a 3-vertex. Let the k -vertex of f_1 be w . Let v be incident a bad $(3, 4, 5)$ -face f_2 with neighbor $(3, 4, 4)$ -face f_3 , and let v have a pendant $(3, 4, 4)$ -face f_4 . Let the 3-vertex of f_4 be x and the 4-vertices of f_4 be y and z (See Figure 5). By the minimality of G , $G \setminus \{f_2, f_3, u, u', x\}$ has a $(1, 1, 0)$ -coloring. Properly color x . If w and x share the same color, then we can properly color u' and u , then properly color v and the 3-vertex of f_2 . Then the coloring can be extended to f_3 by Lemma 7, obtaining a coloring of G . So we can assume that w and x are colored differently. If x is colored by 1 or 2 (w.l.o.g. we may assume that

x is colored by 1), then we can color u' properly and color u by 1. Then we can properly color v and properly color the 3-vertex of f_2 . Finally we can apply Lemma 7 to extend the coloring to f_3 , obtaining a coloring of G . So we can assume that x is colored by 3.

Since x is colored by 3, we may assume that w is colored by 1. Properly color u' and color u by 2. Since x is properly colored, y and z must be colored by 1 and 2. W.l.o.g. let y be colored by 1. Then to avoid being able to re-color x by 1, the two other neighbors of y must be colored 1 and 3. For similar reasons the other two neighbors of z must be colored 2 and 3. Then switch the colors of y and z and color x by 1 or 2 and color v by 3, we can color the 3-vertex of f_2 properly and by Lemma 7, extend the coloring to f_3 , obtaining a coloring of G . \square

Lemma 13. *A $(3, 5, 5)$ -face in G can not have both 5-vertices also be incident to bad $(3, 4, 5)$ -faces and have pendant $(3, 4, 4)$ -faces.*

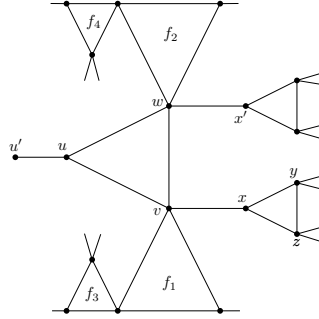


FIGURE 6. Figure for Lemma 13

Proof. Let uvw be a $(3, 5, 5)$ -face in G where $d(v) = d(w) = 5$ and u has pendant neighbor u' . Also let v and w both be incident bad $(3, 4, 5)$ -faces, f_1 and f_2 with neighbor $(3, 4, 4)$ -faces f_3 and f_4 respectively and let v and w have pendant $(3, 4, 4)$ -faces. Let the pendant $(3, 4, 4)$ -faces to v and w have 3-vertices x and x' respectively (See Figure 6). By the minimality of G , $G \setminus \{f_1, f_2, f_3, f_4, u, x, x'\}$ has a $(1, 1, 0)$ -coloring.

Properly color x and x' . If either x or x' has a coloring different from u' , w.l.o.g. we can assume x , then we color u the same as x . We can properly color w and v in that order, then properly color the 3-vertices of f_1 and f_2 . Then by Lemma 7 we can extend the coloring to f_3 and f_4 to obtain a coloring of G . So we can assume that x , x' , and u' are colored the same. If x is colored by 3, since x is properly colored, y and z must be colored by 1 and 2. Then to avoid being able to re-color x by 1, the other two neighbors of y must be colored 1 and 3. For similar reasons the other two neighbors of z must be colored 2 and 3. Then we can switch the colors of y and z and color x differently from u' . Then we follow the above procedure to obtain a coloring for G .

So we may assume that w.l.o.g. x , x' , and u' are all colored by 1. Then we color u by 2 and w by 3. Color the 3-vertex of f_2 properly and by Lemma 7, extend the coloring to f_4 . We now have v adjacent to 3 differently and properly colored vertices. Properly color the outer 4-vertex and the 3-vertex of f_3 in that order, then properly color the 4-vertex of f_1 . If it is colored by 3, then properly color the 3-vertex of f_1 and color v by either 1 or 2 to obtain a coloring of G . If it is not colored by 3, then w.l.o.g. we can assume that it is colored by 1.

Then since it is properly colored, we can color the 3-vertex of f_1 by either 1 or 3 and color v by 2, obtaining a coloring of G . \square

Lemma 14. *A 5-vertex in G that is incident a bad $(3, 4, 5)$ -face and has a pendant $(3, 4, 4)$ -face cannot also be incident a $(4, 4^+, 5)$ -face T_n that is in a (T_0, \dots, T_n) -chain.*

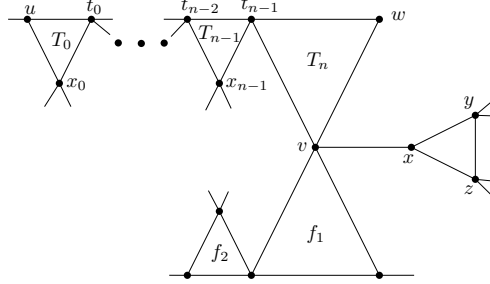


FIGURE 7. Figure for Lemma 14

Proof. Let v be a 5-vertex in G that is incident a bad $(3, 4, 5)$ -face f_1 with neighbor $(3, 4, 4)$ -face f_2 . Let v have a pendant $(3, 4, 4)$ -face with 3-vertex w and 4-vertices y and z . Also let v be incident a $(4, 4^+, 5)$ -face T_n such that there exists a chain of triangles from T_0 to T_n . Let the 4^+ -vertex of T_n be w . Let $S = \{t_i, x_i : 0 \leq i \leq n-1\}$ and let u be the 3-vertex of T_0 (See Figure 7). By the minimality of G , $G \setminus (S \cup \{f_1, f_2, u, x\})$ has a $(1, 1, 0)$ -coloring.

Properly color x . If x and w are colored the same then we can properly color x_{n-1} , t_{n-1} , and v . If $n = 1$, then by Lemma 1, we can color u . If $n \geq 2$, then by Lemma 7 we can extend the coloring to $\{T_0, T_1, \dots, T_{n-1}\}$. Then we can properly color the 3-vertex of f_1 and by Lemma 7 we can extend the coloring to f_2 obtaining a coloring for G . So we can assume that x and w are colored differently.

Let x be colored 1 or 2 and w.l.o.g. we can assume that x is colored by 1. Then we can properly color x_{n-1} and color t_{n-1} by 1. Since w and x are colored differently, either x_{n-1} and t_{n-1} are both colored properly or share the same color. If $n = 1$, then either we can color u properly or we can color u by Lemma 1. If $n \geq 2$, then by Lemma 7 we can extend the coloring to $\{T_0, T_1, \dots, T_{n-1}\}$. Then since t_{n-1} and x are colored the same we can properly color v and the 3-vertex of f_1 . By Lemma 7 we can extend the coloring to f_2 to obtain a coloring of G .

So let x be colored by 3 (then w is colored 1 or 2). Then y and z must be colored by 1 and 2, respectively. To avoid being able to re-color x by 1 or 2, the two other neighbors of y must be colored 1 and 3 and the two other neighbors of z must be colored 2 and 3. Then we switch the colors of y and z and re-color x to be the same as w , and proceed as above to get a coloring for G . \square

Lemma 15. *Every 6-vertex in G that is incident a bad $(3, 4, 6)$ -face can be incident at most two $(3, 4^-, 6)$ -faces.*

Proof. Let v be a 6-vertex in G . Let $vw x$ be a bad $(3, 4, 6)$ -face with $d(w) = 3$ and neighbor $(3, 4, 4)$ -face xyz with 3-vertex y . Let v also be incident non-bad $(3, 4, 6)$ -faces $t_1 t_2 v$ and $u_1 u_2 v$

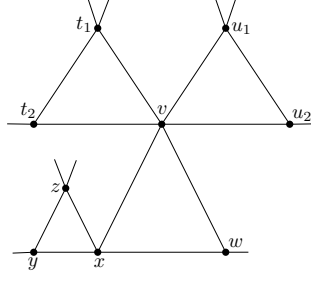


FIGURE 8. Figure for Lemma 15

where $d(t_1) = d(u_1) = 4$ (See Figure 8). By the minimality of G , $G \setminus \{t_1, t_2, u_1, u_2, v, w, x, y, z\}$ has a $(1, 1, 0)$ -coloring. Properly color t_1, t_2, u_1 , and u_2 . If the color set of $\{t_1, t_2, u_1, u_2\}$ is not $\{1, 2, 3\}$, then we can properly color v and w . Then by Lemma 7, we can extend the coloring to x, y , and z , obtaining a coloring of G . So we can assume that the color set of $\{t_1, t_2, u_1, u_2\}$ includes 1, 2, and 3.

If two of $\{t_1, t_2, u_1, u_2\}$ are colored by 3, then we can color z, y , and x properly. If x is colored by 3, then we can color w properly and color v by 1 or 2 to get a coloring of G . If x is colored by 1 or 2, then since x is properly colored we can color w by 3 or the same as x . Then we can color v differently from 3 and x to obtain a coloring of G .

So we can assume that exactly one of vertices in the set $\{t_1, t_2, u_1, u_2\}$ is colored by 3. Then w.l.o.g. we may assume that the color set of $\{t_1, t_2\}$ is $\{1, 3\}$ and the color set of $\{u_1, u_2\}$ is $\{1, 2\}$. Since u_1 and u_2 were colored properly, the outside neighbor of u_2 must be 3. Let u_1 be colored by 1, then since it is colored properly we can recolor u_2 by 1. Then we can color v and w properly, and extend to x, y , and z to obtain a coloring of G . So we can assume that u_1 is colored by 2.

Now color z, y , and x properly in that order. If x is colored by 3 then color w properly. If w is colored by 1, then color v by 2 to get a coloring for G . If w is colored by 2, then since u_1 is colored properly recolor u_2 by 2 and color v by 1 to get a coloring for G . So we can assume that x is colored by 1 or 2. Since x is properly colored we can color w by 3 or the same as x . Then either 1 or 2 but not both is in the color set of $\{x, w\}$. If 1 is in the color set, then v will have only one neighbor colored by 2 so we can color v by 2 and obtain a coloring of G . If 2 is in the color set, then v will have two neighbors colored by 1, but we can recolor u_2 by 2 and color v by 1 to obtain a coloring of G . \square

The following lemma says that a 3-face with k vertices of degree 4 can have at most k chains of triangles ending at it.

Lemma 16. *If a (T_0, T_1, \dots, T_n) -chain and a $(T'_0, T'_1, \dots, T'_m)$ -chain with $T'_m = T_n$ satisfy $T_{n-1} \cap T_n = \{t_n\} = T'_{m-1} \cap T'_m$, then $T_0 = T'_0$.*

Proof. For otherwise, the two chains have a common $(4, 4, 4)$ -face T so that $T = T_a$ and $T = T'_b$. Then we would have a $(T_0, T_1, T_{a-1}, T, T'_{b-1}, \dots, T'_1, T'_0)$ -chain. But by Lemma 8, this chain cannot exist in G . \square

Discharging Procedure

As we mentioned in the introduction, we set the initial charge of a vertex v to be $\mu(v) = 2d(v) - 6$ and the initial charge of a face f to be $\mu(f) = d(f) - 6$. For the discharging procedure we must introduce the notion of a bank, which serves as a temporary placeholder for charges. We set the bank with initial charge zero and will show it has a non-negative final charge.

The following are the rules for discharging:

- (R1) Each 4-vertex gives $\frac{1}{2}$ to each pendant 3-face and the rest to the incident 3-faces evenly.
- (R2) Every 6-vertex gives $\frac{9}{4}$ to incident bad $(3, 4, 6)$ -faces, 2 to other incident $(3, 4^-, 6)$ -faces and $\frac{3}{2}$ to all other incident 3-faces; every 7^+ -vertex gives $\frac{9}{4}$ to all incident 3-faces.
- (R3) Every 6^+ -vertex gives $\frac{1}{2}$ to all pendant 3-faces.
- (R4) Every $(4^+, 4^+, 5^+)$ -face and every $(4, 4, 4)$ -face with a good 4-vertex give $\frac{1}{2}$ to the bank and every bad $(3, 4, 5^+)$ -face gives $\frac{1}{4}$ to the bank.
- (R5) The bank gives $\frac{1}{2}$ to each $(3, 4, 4)$ -face without good 4-vertices.
- (R6) Every 5-vertex gives
 - (a) 2 to each incident $(3, 3, 5)$ -face and $9/4$ to each incident bad $(3, 4, 5)$ -face.
 - (b) $7/4$ to incident non-bad $(3, 4, 5)$ -faces when also incident a bad $(3, 4, 5)$ -face, and gives 2 to incident non-bad $(3, 4, 5)$ -faces otherwise.
 - (c) $5/4$ to incident $(3, 5^+, 5^+)$ -faces when also incident to a bad $(3, 4, 5)$ -face and a pendant $(3, 4^-, 4^-)$ -face, and gives $3/2$ to incident $(3, 5^+, 5^+)$ -faces otherwise.
 - (d) $3/2$ to all $(4, 4^+, 5)$ -faces with a chain of triangles to a $(3, 4, 4)$ -face and gives 1 to $(4, 4^+, 5)$ -faces otherwise.
 - (e) $1/2$ to each pendant $(3, 4^-, 4^-)$ -face and $(3, 3, k)$ -face and $1/4$ to all other pendant 3-faces.

Let v be a k -vertex. By Proposition 1, $k \geq 3$.

For $k = 3$, the final charge $\mu^*(v)$ of v is $\mu^*(v) = \mu(v) = 0$.

For $k = 4$, by (R1), the final charge of v is 0. We note that v gives at least 1 to each incident 3-face, and gives at least $3/2$ to 3-faces when v is a good 4-vertex.

For $k = 5$, if v has at most one incident 3-face, then by (R6a) and (R6e), $\mu^*(v) \geq \mu(v) - \frac{9}{4} \cdot 1 - \frac{1}{2} \cdot 3 = 1/4 > 0$. Let v have two incident 3-faces f_1 and f_2 and a pendant 3-face f_3 .

Let f_3 be a $(3, 4^-, 4^-)$ -face. When f_1 is a bad $(3, 4, 5)$ -face, by Lemma 3 f_2 cannot be a $(3, 4^-, 5)$ -face. By Lemma 14, if f_2 is a $(4, 4^+, 5)$ -face, then there is no chain of triangles from some $(3, 4, 4)$ -face to f , so by (R6a), (R6c), (R6d), and (R6e), $\mu^*(v) \geq \mu(v) - \frac{1}{2} \cdot 1 - \frac{9}{4} \cdot 1 - \frac{5}{4} \cdot 1 = 0$. When f_1 is a non-bad $(3, 4, 5)$ -face, then by Lemma 3, f_2 cannot be a $(3, 4^-, 5)$ -face, so by (R6b), (R6c), (R6d), and (R6e), $\mu^*(v) \geq \mu(v) - \frac{1}{2} \cdot 1 - 2 \cdot 1 - \frac{3}{2} \cdot 1 = 0$. When neither f_1 nor f_2 are $(3, 4^-, 5)$ -faces, by (R6c), (R6d), and (R6e), $\mu^*(v) \geq \mu(v) - \frac{1}{2} \cdot 1 - \frac{3}{2} \cdot 2 = \frac{1}{2} > 0$.

Now let f_3 be a $(3, 4, 5)$ -face. When f_1 or f_2 is $(3, 4^-, 5)$ -face, by Lemma 3, the other one cannot be a $(3, 4^-, 5)$ -face, so by (R6b), (R6c), (R6d), and (R6e), $\mu^*(v) \geq \mu(v) - \frac{1}{4} \cdot 1 - \frac{9}{4} \cdot 1 - \frac{3}{2} \cdot 1 = 0$. When neither f_1 nor f_2 are $(3, 4^-, 5)$ -faces, by rules (R6c), (R6d), and (R6e), $\mu^*(v) \geq \mu(v) - \frac{1}{4} \cdot 1 - \frac{3}{2} \cdot 2 = \frac{3}{4} > 0$.

Finally, let v have two incident 3-faces f_1 and f_2 , and no pendant 3-face. If f_1 is a bad $(3, 4, 5)$ -face, then by Lemma 11, f_2 cannot also be a bad $(3, 4, 5)$ -face or a $(3, 3, 5)$ -face.

Then by (R6), $\mu^*(v) \geq \mu(v) - \frac{9}{4} \cdot 1 - \frac{7}{4} \cdot 1 = 0$. If neither f_1 nor f_2 is a bad $(3, 4, 5)$ -face, then by (R6b), (R6c), and (R6d), $\mu^*(v) \geq \mu(v) - 2 \cdot 2 = 0$.

For $k = 6$, if v is incident to at most two 3-faces, then by (R2) and (R3), $\mu^*(v) \geq \mu(v) - \frac{9}{4} \cdot 2 - \frac{1}{2} \cdot 2 = \frac{1}{2}$. So we can assume that v is incident to three 3-faces. If v is incident a bad $(3, 4, 6)$ -face then by Lemma 15 only one other incident 3-face can be a $(3, 4^-, 6)$ -face. So by (R2), $\mu^*(v) \geq \mu(v) - \frac{9}{4} \cdot 2 - \frac{3}{2} \cdot 1 = 0$. If v is not incident a bad $(3, 4, 6)$ -face, then by (R2), $\mu^*(v) \geq \mu(v) - 2 \cdot 3 = 0$.

For $k \geq 7$, if k is odd, then $\mu^*(v) \geq \mu(v) - \frac{k-1}{2} \cdot \frac{9}{4} - \frac{1}{2} \cdot 1 = 2k - 6 - \frac{9k-9}{8} - \frac{4}{8} = \frac{7k-43}{8} \geq \frac{3}{4}$. If k is even, then $\mu^*(v) \geq \mu(v) - \frac{k}{2} \cdot \frac{9}{4} = 2k - 6 - \frac{9k}{8} = \frac{7k-48}{8} \geq 1$.

Now let f be a k -face. Since G is a simple graph, $k \geq 3$. By the condition that there is no 4-cycle and 5-cycle, $k = 3$ or $k \geq 6$. Since no faces above degree 3 are involved in the discharging procedure, the final charge of 6^+ -face f is $\mu^*(f) = \mu(f) = d(f) - 6 \geq 0$.

For $k = 3$, by Lemma 2, we have no $(3, 3, 4^-)$ -faces, but we still have a few different cases:

Case 1: Face f is a $(3, 3, 5^+)$ -face. By Lemma 4, f will have two pendant neighbors of degree 4 or higher. So by (R1), (R2), (R4), and (R7), $\mu^*(f) \geq (3 - 6) + 2 \cdot 1 + \frac{1}{2} \cdot 2 = 0$.

Case 2: Face f is a $(3, 4, 4)$ -face. By Lemma 5, f will have a pendant neighbor of degree 4 or higher. If f has a good 4-vertex, then by (R1), $\mu^*(f) \geq \mu(f) + \frac{3}{2} \cdot 1 + 1 \cdot 1 + \frac{1}{2} \cdot 1 = 0$. If f has no good 4-vertices, then by (R5), f receives $1/2$ from the bank, so $\mu^*(f) = \mu(f) + 1 \cdot 2 + \frac{1}{2} \cdot 1 + \frac{1}{2} = 0$.

Case 3: Face f is a bad $(3, 4, 5)$ -face. By (R1), (R4) and (R6a), $\mu^*(f) = \mu(f) + 1 \cdot 1 + \frac{9}{4} \cdot 1 - \frac{1}{4} \cdot 1 = 0$.

Case 4: Face f is a non-bad $(3, 4, 5)$ -face. If the 5-vertex of f is not incident a bad $(3, 4, 5)$ -face, then by (R1) and (R6b), $\mu^*(f) = \mu(f) + 1 \cdot 1 + 2 \cdot 1 = 0$. If the 5-vertex of f is incident a bad $(3, 4, 5)$ -face, then by Lemma 12, f has a pendant neighbor of degree 4 or higher. So by (R1), (R6b), and (R6e), $\mu^*(f) \geq \mu(f) + 1 \cdot 1 + \frac{7}{4} \cdot 1 + \frac{1}{4} \cdot 1 = 0$.

Case 5: Face f is a $(3, 4, 6)$ -face. If f is a bad $(3, 4, 6)$ -face, then by (R1), (R2), and (R4), $\mu^*(f) = \mu(f) + 1 \cdot 1 + \frac{9}{4} \cdot 1 - \frac{1}{4} \cdot 1 = 0$. If f is a non-bad $(3, 4, 6)$ -face then by (R1) and (R2), $\mu^*(f) = \mu(f) + 1 \cdot 1 + 2 \cdot 1 = 0$.

Case 6: Face f is a $(3, 4, 7^+)$ -face. By (R1) and (R2), $\mu^*(f) = \mu(f) + 1 \cdot 1 + \frac{9}{4} \cdot 1 = \frac{1}{4}$.

Case 7: Face f is a $(3, 5, 5)$ -face. If neither 5-vertex of f is also incident to a bad $(3, 4, 5)$ -face and a pendant $(3, 4^-, 4^-)$ -face, then by (R6c), $\mu^*(f) = \mu(f) + \frac{3}{2} \cdot 2 = 0$. If one of the 5-vertices of f is also incident to a bad $(3, 4, 5)$ -face and a pendant $(3, 4^-, 4^-)$ -face then by Lemma 12, f must have a pendant neighbor of degree 4 or higher. In addition, by Lemma 13 the other 5-vertex of f cannot have both an incident bad $(3, 4, 5)$ -face and a pendant $(3, 4^-, 4^-)$ -face. So by (R6c) and (R6e), $\mu^*(f) = \mu(f) + \frac{5}{4} \cdot 1 + \frac{1}{4} \cdot 1 + \frac{3}{2} \cdot 1 = 0$.

Case 8: Face f is a $(3, 5, 6^+)$ -face. If the 5-vertex of f is not incident to a bad $(3, 4, 5)$ -face and a pendant $(3, 4^-, 4^-)$ -face then by (R2) and (R6c), $\mu^*(f) \geq \mu(f) + \frac{3}{2} \cdot 2 = 0$. If the 5-vertex of f has both an incident bad $(3, 4, 5)$ -face and a pendant $(3, 4^-, 4^-)$ -face, then by Lemma 12 f must have a pendant neighbor of degree 4 or higher. So by (R2), (R6c), and (R6e), $\mu^*(f) \geq \mu(f) + \frac{5}{4} \cdot 1 + \frac{1}{4} \cdot 1 + \frac{3}{2} \cdot 1 = 0$.

Case 9: Face f is a $(3, 6^+, 6^+)$ -face. By (R2), $\mu^*(f) \geq \mu(f) + \frac{3}{2} \cdot 2 = 0$.

Case 10: Face f is a $(4, 4, 4)$ -face. If f has no good 4-vertices then by (R1), $\mu^*(f) = \mu(f) + 1 \cdot 3 = 0$. If f has a good 4-vertex then by (R1) and (R4), $\mu^*(f) \geq \mu(f) + 1 \cdot 2 + \frac{3}{2} \cdot 1 - \frac{1}{2} \cdot 1 = 0$.

Case 11: Face f is a $(4^+, 4^+, 5^+)$ -face. If f has no chains of triangles to a $(3, 4, 4)$ -face, then each incident vertex gives at least 1 to f , so $\mu^*(f) \geq \mu(f) + 1 \cdot 3 = 0$. If f has a chain of triangles to a $(3, 4, 4)$ -face then by (R6d), at least one vertex must give $\frac{3}{2}$ to f , so combined with (R4), $\mu^*(v) \geq \mu(v) + 1 \cdot 2 + \frac{3}{2} \cdot 1 - \frac{1}{2} \cdot 1 = 0$.

Finally, we show that the bank has a non-negative charge. By Lemma 10, for each $(3, 4, 4)$ -face without good 4-vertices in G , there exist at least two chains of triangles from the $(3, 4, 4)$ -face to a bad $(3, 4, 5^+)$ -face, a $(4, 4, 4)$ -face with a good 4-vertex, or a $(4^+, 4^+, 5^+)$ -face. Then by Lemma 16, there exist at most two chains of triangles to $(4^+, 4^+, 5^+)$ -face from $(3, 4, 4)$ -faces and at most one chain of triangles to a $(3, 4, 5^+)$ -face from $(3, 4, 4)$ -faces. So we can see the transfer of charge from triangles with extra charge to the bank and back to $(3, 4, 4)$ -faces is a transfer of $\frac{1}{4}$ charge over each chain of triangles. Each $(4, 4, 4)$ -face with a good 4-vertex and $(4^+, 4^+, 5^+)$ -face gives $\frac{1}{2}$ to the bank, and the bank will give at most $\frac{1}{4} \cdot 2$ to $(3, 4, 4)$ -faces for each $(4, 4, 4)$ -face with a good 4-vertex or $(4^+, 4^+, 5^+)$ -face. Also, each bad $(3, 4, 5^+)$ -face gives $\frac{1}{4}$ to the bank, and the bank will give at most $\frac{1}{4} \cdot 1$ to $(3, 4, 4)$ -faces for each bad $(3, 4, 5^+)$ -face. Hence the bank will always have a non-negative charge.

This completes the discharging, showing that the final charges of all faces, vertices, and the bank are non-negative, a contradiction to (1). This completes the proof of Theorem 1.1.

3. $(3, 0, 0)$ -COLORING OF PLANAR GRAPHS

In this section, we give a proof for Theorem 2. Our proof will again use a discharging method. Let G be a minimum counterexample to Theorem 2, that is, G is a planar graph without 4-cycles and 5-cycles and is not $(3, 0, 0)$ -colorable, but any proper subgraph of G is properly $(3, 0, 0)$ -colorable. We may assume that vertices colored by 1 may have up to three neighbors colored by 1.

The following is a very useful tool to extend a coloring on a subgraph of G to include more vertices.

Lemma 17. *Let H be a proper subgraph of G . Given a $(3, 0, 0)$ -coloring of $G - H$, if two neighbors of $v \in H$ are colored so that one is a 5^- -vertex and the other is nicely colored, then the coloring can be extended to $G - (H - v)$ such that v is nicely colored by 1.*

Proof. Let H be a subgraph of G such that $G - H$ has a $(3, 0, 0)$ -coloring. Let $v \in H$ have neighbors u and w that are colored. Let $d(u) \leq 5$ and let w be nicely colored. Color v by 1. Since w is nicely colored, if this coloring is invalid, then u must be colored by 1. In addition, u must have at least 3 neighbors colored by 1. To avoid recoloring u by 2 or 3, u must have at least one neighbor of color 2 and at least one neighbor of color 3. This implies that $d(u) \geq 6 > 5$, a contradiction. So v is colorable by 1. In addition, since the deficiency of color 1 is 3 and v only has 2 neighbors, it follows that v is nicely colored. \square

Lemma 18. *Every 3-vertex in G has a 6^+ -vertex as a neighbor.*

Proof. Let v be a vertex in G such that each neighbor vertex of v has degree 5. By the minimality of G , $G - v$ is $(3, 0, 0)$ -colorable. If two vertices in $N(v)$ share the same color, then v can be properly colored, so we can assume all the neighbors of v are colored differently. Let u be the neighbor of v that is colored by 1. Then u must have 3 neighbors colored by 1 to forbid v to be colored by 1. In addition, u must have neighbors colored by 2 and 3 to forbid v to be colored by 2 or 3. Then, u has at least 6 neighbors, a contradiction. \square

Let a $(3, 3, 3^+)$ -face to be *poor* if the pendant neighbors of the two 3-vertices have degrees at most 5. A $(3, 3^+, 3^+)$ -face is *semi-poor* if exactly one of the pendant neighbors of the 3-vertices has degree 5 or less. A 3-face is *non-poor* if each 3-vertex on it has the pendant neighbor being a 6^+ -vertex. Finally, a *poor 3-vertex* is a 3-vertex on a poor or semi-poor 3-face that has a 5^- -vertex as its pendant neighbor.

Lemma 19. *All $(3, 3, 6^-)$ -faces in G are non-poor.*

Proof. For all $(3, 3, 5^-)$ -faces in G , the proof is trivial by Lemma 18. Let uvw be a $(3, 3, 6^-)$ -face in G with $d(u) = d(v) = 3$ such that the pendant neighbor v' of v has degree at most 5. By the minimality of G , $G \setminus \{u, v\}$ is $(3, 0, 0)$ -colorable. Properly color u and color v differently than both w and v' . Then u and v are both colored by 2 or 3, w.l.o.g. assume 2. This means that u' and v' share the same color (where u' is the pendant neighbor of u), different from the color of w .

Let w be colored by 1, then to avoid being able to recolor u or v by 1, w must have 3 outer neighbors colored by 1. Then w can be recolored by 2 or 3 depending on the color of its fourth colored neighbor. We recolor w by 2 or 3 and recolor u and v by 1 to get a coloring of G , a contradiction.

So we may assume that w is colored by 3, and that u' and v' are colored by 1. To avoid recoloring v by 1, v' must have at least 3 neighbors colored by 1. In addition, to avoid recoloring v' by 2 or 3 and coloring v by 1, v' must have neighbors colored by both 2 and 3. This contradicts that v' has degree less than 6. \square

Lemma 20. *No vertex $v \in V(G)$ can have $\lfloor \frac{d(v)}{2} \rfloor$ incident poor 3-faces.*

Proof. Let v be a k -vertex in G with $\lfloor \frac{k}{2} \rfloor$ incident poor $(3, 3, k)$ -faces. Let u_1, u_2, \dots, u_k be the neighbors of v , and let u'_i be the pendant neighbor if u_i is in a poor 3-face. Note that $d(u'_i) \leq 5$ and we know that all except possibly u_k are in poor 3-faces.

By the minimality of G , $G \setminus \{v, u_1, u_2, \dots, u_{k-1}\}$ is $(3, 0, 0)$ -colorable. If $d(v)$ is odd, then by Lemma 17, for all i with $1 \leq i \leq k-1$, we can color u_i by 1. Then we can properly color v to get a coloring of G , so we can assume that $d(v)$ is even. If $d(v)$ is even, then by Lemma 17, for all i with $1 \leq i \leq k-2$, we can color u_i by 2. Then if u_k is colored by 1 we can color u_{k-1} properly and v properly to get a coloring of G . If u_k is colored by 2 or 3, then it is colored properly and by Lemma 17 we can color u_{k-1} by 1. Then we can properly color v to get a coloring of G , a contradiction. \square

Lemma 21. *If an 8-vertex v is incident to three incident poor $(3, 3, 8)$ -faces, then it cannot be incident to a semi-poor face, nor two pendant 3-faces.*

Proof. Let v be an 8-vertex in G with 3 incident poor $(3, 3, 8)$ -faces. Let u_1, u_2, \dots, u_6 be the 3-vertices in the poor $(3, 3, 8)$ -face and let u'_1, u'_2, \dots, u'_6 be the corresponding pendant neighbors, respectively. We know that for all i with $1 \leq i \leq 6$, $d(u'_i) \leq 5$.

(i) Let vu_7u_8 be the incident semi-poor face with u_7 being the poor 3-vertex. Then by the minimality of G , $G \setminus \{v, u_1, u_2, \dots, u_7\}$ is $(3, 0, 0)$ -colorable. By Lemma 17, u_1, u_2, \dots, u_6 can be colored by 1. Then if u_8 is colored by 1, we can properly color u_7 and then v to get a coloring of G . So we may assume that u_8 is not colored by 1, in which case it is nicely colored and we may color u_7 with 1 by Lemma 17, and then properly color v to get a coloring of G , a contradiction.

(ii) Let u_7 and u_8 be the bad 3-vertices adjacent to v . Then $G \setminus \{v, u_1, u_2, \dots, u_7, u_8\}$ is $(3, 0, 0)$ -colorable, by the minimality of G . Properly color both u_7 and u_8 . If either u_7 or u_8 is colored by 1 or both have the same color, then by Lemma 17, we may color u_1, u_2, \dots, u_6 by 1 and then properly color v . So we may assume that u_7 is colored by 2 and u_8 is colored by 3. Then we properly color u_1, u_2, \dots, u_6 , and it follows that for each i with $1 \leq i \leq 3$, u_{2i-1} and u_{2i} must be colored differently. Then v can have at most 3 neighbors colored by 1, all properly colored, so v can be colored by 1, a contradiction. \square

Lemma 22. *If a 7-vertex v is incident to two poor $(3, 3, 7)$ -faces, then it cannot be (i) incident to a semi-poor $(3, 6^-, 7)$ -face and adjacent to a pendant 3-face, or (ii) adjacent to three pendant 3-faces.*

Proof. Let v be a 7-vertex in G with 2 incident poor $(3, 3, 7)$ -faces. Let u_1, u_2, u_3 , and u_4 be the 3-vertices on the poor $(3, 3, 7)$ -faces and let u'_1, u'_2, u'_3 , and u'_4 be their corresponding pendant neighbors, respectively. We know that for all i with $1 \leq i \leq 4$, $d(u'_i) \leq 5$.

(i) Let vu_5u_6 be a semi-poor face with u_5 being a poor 3-vertex and $d(u_6) \leq 6$ and let u_7 be a bad 3-vertex adjacent to v . By the minimality of G , $G \setminus \{v, u_1, u_2, u_3, u_4, u_5, u_7\}$ is $(3, 0, 0)$ -colorable. Since at this point u_6 has only 4 colored neighbors, if u_6 is colored by 1 then either it is nicely colored or it can be recolored properly. If u_6 is not nicely colored, then recolor u_6 properly.

Color u_7 properly. If u_7 is colored by 1, then by Lemma 17, we can color u_1, u_2, \dots, u_5 by 1 and then color v properly, a contradiction. So we may assume w.l.o.g. that u_7 is colored by 2. Color u_1, u_2, \dots, u_5 properly. Then, for each i with $1 \leq i \leq 3$, u_{2i} and u_{2i-1} are colored differently and nicely. This leaves v with at most 3 neighbors colored by 1, all nicely, so we may color v by 1 to get a coloring of G , a contradiction.

(ii) Let u_5 , u_6 , and u_7 be the bad 3-vertices adjacent to v . By the minimality of G , $G \setminus \{v, u_1, \dots, u_7\}$ is $(3, 0, 0)$ -colorable. Properly color u_5 , u_6 , and u_7 . If the set $\{u_5, u_6, u_7\}$ does not contain both colors 2 and 3, then by Lemma 17, we can color u_1, u_2, u_3 , and u_4 by 1 and color v properly. So we can assume that $\{u_5, u_6, u_7\}$ contains both colors 2 and 3. This implies that at most one vertex is colored by 1. So we properly color u_1, u_2, u_3 , and u_4 . Then v has at most 3 neighbors colored by 1, all nicely, so we can color v by 1 to get a coloring of G , a contradiction. \square

Lemma 23. *Let uvw be a semi-poor $(3, 7, 7)$ -face in G such that $d(v) = d(w) = 7$. Then vertices v and w cannot both be 7-vertices that are incident to two poor 3-faces, one semi-poor $(3, 7, 7)$ -face, and adjacent to one pendant 3-face.*

Proof. Let uvw be a semi-poor $(3, 7, 7)$ -face in G such that $d(v) = d(w) = 7$ and both v and w are incident to two poor 3-faces, one $(3, 7, 7)$ -face, and adjacent to one pendant 3-face. Let the neighbors of v and w be t_1, t_2, \dots, t_5 and z_1, z_2, \dots, z_5 , respectively such that t_5 and z_5 are bad 3-vertices (See Figure 9).

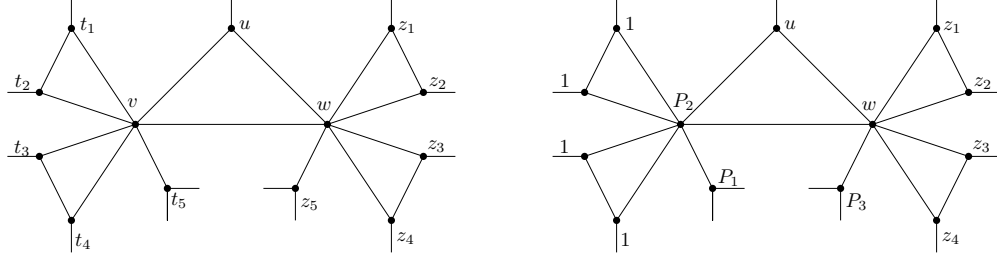


FIGURE 9. Figure for Lemma 23

By the minimality of G , $G \setminus \{u, v, w, t_1, t_2, \dots, t_5, z_1, z_2, \dots, z_5\}$ is $(3, 0, 0)$ -colorable. By Lemma 17, we can color t_1, t_2, t_3 , and t_4 by 1. Then properly color t_5 , v , and z_5 in that order. Vertex v will not be colored by 1, so w.l.o.g. let's assume that v is properly colored by 2. If z_5 is colored by 1, then by Lemma 17, we can color z_1, z_2, z_3, z_4 , and u by 1 and then properly color w , to get a coloring of G , a contradiction. So we can assume that z_5 is not colored by 1. Then we properly color z_1, z_2, z_3, z_4 and u , so w can have at most 3 neighbors colored by 1, all properly. We can color v by 1 to get a coloring of G , a contradiction. \square

Discharging Procedure:

We start the discharging process now. Recall that the initial charge for a vertex v is $\mu(v) = 2d(v) - 6$ and the initial charge for a face f is $\mu(f) = d(f) - 6$.

We introduce the following discharging rules:

- (R1) Every 4-vertex gives 1 to each incident 3-face.
- (R2) Every 5 and 6-vertex gives 2 to each incident 3-face.
- (R3) every 6^+ -vertex gives 1 to each adjacent pendant 3-face.
- (R4) Each d -vertex with $7 \leq d \leq 10$ gives 3 to each incident poor $(3, 3, *)$ -face, 2 to each incident semi-poor 3-face, except 7-vertices give 1 to special semi-poor 3-face, where a special semi-poor $(3, 7, 7+)$ -face is a semi-poor 3-face incident to a 7-vertex which is also incident to two poor 3-faces and adjacent to one pendant 3-face. Each d -vertex with $7 \leq d \leq 10$ gives 1 to all other incident 3-faces.
- (R5) Every 11^+ -vertex gives 3 to all incident 3-faces.

Now let v be a k -vertex. By Proposition 1, $k \geq 3$.

When $k = 3$, v is not involved in the discharging process, so $\mu^*(v) = \mu(v) = 0$.

When $k = 4$, by Proposition 1, v can have at most 2 incident 3-faces. By (R1), $\mu^*(v) \geq \mu(v) - 1 \cdot 2 = 0$.

When $k = 5$, by Proposition 1, v can have at most 2 incident 3-faces. By (R2), $\mu^*(v) \geq \mu(v) - 2 \cdot 2 = 0$.

When $k = 6$, by Proposition 1, v can have $\alpha \leq 3$ incident 3-faces, and at most $(k - 2\alpha)$ pendant 3-faces. By (R2) and (R3), $\mu^*(v) \geq \mu(v) - 2 \cdot \alpha - 1 \cdot (k - 2\alpha) = k - 6 = 0$.

When $k = 7$, v has an initial charge $\mu(v) = 7 \cdot 2 - 6 = 8$. By Lemma 20, v has at most two poor 3-faces. If v has less than two incident poor 3-faces, then by (R3) and (R4), $\mu^*(v) \geq \mu(v) - 3 \cdot 1 - 1 \cdot 5 = 0$ since v gives at most one charge per vertex excluding vertices in poor 3-faces. So assume that v has exactly 2 incident poor 3-faces. By Lemma 22, v is

adjacent to at most two pendant 3-faces, and if it is incident to a semi-poor $(3, 6^-, 7)$ -face, then v is not adjacent to a pendant 3-face. So if v is not incident to a semi-poor $(3, 7^+, 7)$ -face, then by (R3) and (R4), $\mu^*(v) \geq \mu(v) - 3 \cdot 2 - 2 \cdot 1 = 0$; If v is incident to a semi-poor $(3, 7^+, 7)$ -face, then by rules (R3) and (R4), $\mu^*(v) \geq \mu(v) - 3 \cdot 2 - 1 \cdot 1 - 1 \cdot 1 = 0$.

When $k = 8$, v has an initial charge $\mu(v) = 8 \cdot 2 - 6 = 10$. By Lemma 20, v has at most three poor 3-faces. If v has less than 3 incident poor 3-faces, then by (R3) and (R4), $\mu^*(v) \geq \mu(v) - 3 \cdot 2 - 1 \cdot 4 = 10 - 6 - 4 = 0$ since v gives at most one charge per vertex excluding vertices in poor 3-faces. So let v is incident to exactly 3 poor 3-faces. By Lemma 21, v cannot be incident to a semi-poor 3-face or adjacent to two pendant 3-faces, then $\mu^*(v) \geq \mu(v) - 3 \cdot 3 - 1 \cdot 1 = 0$.

When $k = 9$, by Lemma 20, v is incident to at most three poor 3-faces. The worst case occurs when v is incident 3 poor $(3, 3, 9)$ -faces, incident one semi-poor $(3, 3, 9)$ -face, and pendant one 3-face. So by (R3) and (R4), $\mu^*(v) \geq \mu(v) - 1 \cdot 1 - 3 \cdot 3 - 2 \cdot 1 = 12 - 1 - 9 - 2 = 0$.

When $k = 10$, by Lemma 20, v is incident to at most four poor $(3, 3, 10)$ -faces. So by (R3) and (R4), $\mu^*(v) \geq \mu(v) - 3 \cdot 4 - 2 \cdot 1 = 14 - 3 \cdot 4 - 2 \cdot 1 = 0$.

When $k \geq 11$, we assume that v is incident to α 3-faces, then by Proposition 1, $\alpha \leq \lfloor k/2 \rfloor$. Thus the final charge of v is $\mu^* \geq 2k - 6 - 3\alpha - 1 \cdot (k - 2\alpha) = k - \alpha - 6 \geq 0$.

Now let f be a k -face in G . By the conditions on G , $k = 3$ or $k \geq 6$. When $k \geq 6$, f is not involved in the discharging procedure, so $\mu^*(f) = \mu(f) = k - 6 \geq 0$. So in the following we only consider 3-faces.

Case 1: f is a $(4^+, 4^+, 4^+)$ -face. By the rules, each 4^+ -vertex on f gives at least 1 to f , so $\mu^*(f) \geq \mu(f) + 1 \cdot 3 = 0$.

Case 2: f is a $(3, 4^+, 4^+)$ -face with vertices u, v, w such that $d(u) = 3$. If u is not a poor 3-vertex, then by (R2), f gains 1 from the pendant neighbor of u and by the other rules, f gains at least 2 from vertices on f , thus $\mu^*(f) \geq \mu(f) + 1 \cdot 3 = 0$. If u is a poor vertex (it follows that f is a semi-poor 3-face), then by Lemma 18, f is a $(3, 4^+, 6^+)$ -face. Since v or w is a 6^+ -vertex, it gives at least 2 to f unless f is a special semi-poor $(3, 7, 7^+)$ -face, and as the other is a 4^+ -vertex, it gives at least 1 to f . Therefore, if f is not a special semi-poor 3-face, then $\mu^*(f) \geq \mu(f) + 2 \cdot 1 + 1 \cdot 1 = 0$; if f is a special semi-poor $(3, 7, 8^+)$ -face, then f receives at least 2 from the 8^+ -vertex, so $\mu^*(f) \geq \mu(f) + 2 \cdot 1 + 1 \cdot 1 = 0$. If f is a special semi-poor $(3, 7, 7)$ -face so that both v and w are incident to two poor 3-faces, one semi-poor $(3, 7, 7)$ -face and adjacent to one pendant 3-face, then by Lemma 23, is impossible.

Case 3: f is a $(3, 3, 4^+)$ -face with 4^+ -vertex v . If $d(v) \geq 11$, then by (R5), $\mu^*(f) \geq \mu(f) + 3 = 0$. So assume $d(v) \leq 10$. By Lemma 18, if $4 \leq d(v) \leq 6$, then each 3-vertex has the pendant neighbor of degree 6 or higher. So by (R1) and (R3) (when $d(v) = 4$), $\mu^*(f) \geq \mu(f) + 1 \cdot 3 = 0$, or by (R1) and (R2) (when $d(v) > 4$), $\mu^*(f) = \mu(f) + 2 \cdot 1 + 1 \cdot 1 = 0$.

Let $7 \leq d(v) \leq 10$. If f is poor, then by (R4), $\mu^*(f) = \mu(f) + 3 \cdot 1 = 0$. If f is semi-poor, then one 3-vertex on f is adjacent to a 6^+ -vertex and thus by (R3) f gains 1 from it, together the 2 that f gains from v by (R4), we have $\mu^*(f) = \mu(f) + 2 \cdot 1 + 1 \cdot 1 = 0$. If f is non-poor, then both 3-vertices on f are adjacent to the pendant neighbors of degrees more than 5, thus by (R3) and (R4), $\mu^*(f) = \mu(f) + 1 \cdot 2 + 1 \cdot 1 = 0$.

Case 4: f is a $(3, 3, 3)$ -face. By Lemma 18, each 3-vertex will have the pendant neighbor of degree 6 or higher, so by (R3), $\mu^*(f) = \mu(f) + 1 \cdot 3 = 0$.

Since for all $x \in V \cup F$, $\mu^*(x) \geq 0$, $\sum_{v \in V} \mu^*(v) + \sum_{f \in F} \mu^*(f) \geq 0$, a contradiction. This completes the proof of Theorem 1.2.

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